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Fracture mechanical assessment of interface cracks with contact zones in piezoelectric bimaterials under thermoelectromechanical loadings I. Electrically permeable interface cracks

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Abstract

An electrically permeable interface crack with a frictionless contact zone at the right crack tip between two semi-infinite piezoelectric spaces under the action of a remote electromechanical loading and a temperature flux is considered. Assuming that all fields are independent on the coordinate x_2 co-directed with the crack front, the stresses, the electrical and the temperature fluxes as well as the derivatives of the jumps of the displacements, the electrical potential and the temperature at the interface are presented via a set of analytic functions in the (x_1, x_3)-plane with a cut along the crack. Due to this representation firstly an auxiliary problem concerning the direction of the heat flux permitting a transition from a perfect thermal contact to a separation has been solved for a piezoelectric bimaterial. Besides, an inhomogeneous combined Dirichlet–Riemann boundary value problem has been formulated and solved exactly for the above mentioned interface crack. Stress and electrical displacements intensity factors are found in a clear analytical form which is especially easier for a small contact zone length. A simple equation and a closed form analytical formula for the determination of the real contact zone length have been derived and compared with the associated equation of the classical (oscillating) interface crack model defining the zone of crack faces interpenetration. For a numerical illustration of the obtained results a bimaterial cadmium selenium/glass has been used, and the influence of the heat flux upon the contact zone length and the associated stress intensity factor has been shown.

Keywords: Electrically permeable interface crack; Contact zone; Piezoelectric bimaterial; Heat flux

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1. Introduction

In the last years piezoelectric bimaterials are widely used in different devices working under a high-temperature environment. However, piezoceramic bimaterials often contain various microdefects and particularly interface cracks. Such cracks are the most dangerous kind of defects especially under an essential temperature field. The problem of an interface crack in a piezoelectric bimaterial is rather complicated under pure electromechanical loading even and there are only several publications related to this subject. The consideration of a temperature field in addition to the electromechanical loading leads to an essential enlargement of the associated mathematical model and therefore the number of results obtained till now do not correspond to the importance of the problem in question.

An interface crack in an infinite piezoelectric bimaterial under the action of a remote temperature flux has been analytically investigated in the paper by Shen and Kuang (1998), where the representations of Lekhnitskii (1963), Eshelby et al. (1953) and Stroh (1958) extended for the piezoelectric case by Barnett and Lothe (1975) have been used. Thereby an electrically impermeable crack has been assumed in this paper. Later a similar problem for an electrically permeable interface crack has been considered in the paper by Gao and Wang (2001), where also as in the paper by Shen and Kuang (1998) the classical interface crack model has been used. The solutions obtained in the frame of this model possesses the oscillating singularities at the crack tips which was found by Williams (1959). Nevertheless, for small zone lengths of overlapping of crack faces this solution is rather useful for an interface crack investigation because for such cases the required fracture mechanical parameters can be accurately defined by this solution. However, the existence of an essential shear loading and a temperature field lead in certain cases to the appearance of a long contact zone of the interface crack faces. In such cases the approach based upon the initial assumption concerning the existence of a contact zone (Comninou, 1977) should be used.

A penny-shaped interface crack with a contact region between two isotropic materials under a thermomechanical loading has been investigated by Martin-Moran et al. (1983) and by Barber and Comninou (1983) by means of the method of singular integral equations. The thermal conditions in the zone of the mechanical contact of the crack faces in particular have been investigated in these papers, and important conclusions concerning the formulation of these conditions depending on the direction of the heat flux have been developed. An interface crack with a contact zone in an anisotropic bimaterial under thermomechanical loading has been analytically studied by Herrmann and Loboda (2001), where a method similar to the approach of the present paper has been developed. A thermopiezoelectric bimaterial with an interface crack under the assumption of a contact zone model has been investigated by Qin and Mai (1999) by means of Lekhnitskii–Eshelby–Stroh formalism. The method of singular integral equations has been used in this paper, and the crack faces including the contact zones were assumed to be thermally and electrically insulated.

In the present investigation an exact analytical solution for a crack with a contact zone between two piezoelectric semi-infinite spaces under a remote electromechanical loading and a temperature flux has been found. A transcendental equation and rather simple formula for the determination of the contact zone length as well as closed formulas for the associated stress and electrical intensity factors have been found. In this part of the paper the attention is focused on the case of an electrically permeable crack, while the electrically impermeable will be considered in Part II of this paper.

2. Basic relations for a thermopiezoelectric solid

For a stationary process in the absence of body forces and free charges the constitutive relations for a linear piezothermoelectric material can be presented according to Mindlin (1974) in the form

$$\Pi_{iJ} = E_{iJKl} V_{K,l} - \beta_{iJ} T, \quad \Pi_{iJ,i} = 0 \tag{1}$$

$$q_i = -\lambda_{ij} T_{,j}, \quad q_{i,i} = 0 \tag{2}$$

where

$$V_K = \begin{cases} u_k, & K = 1,2,3 \\ \varphi, & K = 4 \end{cases} \tag{3}$$

$$\Pi_{iJ} = \begin{cases} \sigma_{ij}, & i,J = 1,2,3 \\ D_i, & i = 1,2,3;\ J = 4 \end{cases} \tag{4}$$

and

$$E_{iJKl} = \begin{cases} c_{ijkl}, & J,K = 1,2,3 \\ e_{lij}, & J = 1,2,3;\ K = 4 \\ e_{ikl}, & K = 1,2,3;\ J = 4 \\ -\varepsilon_{il}, & J = K = 4 \end{cases} \tag{5}$$

In the relations (1)–(5) u_k, φ, σ_{ij}, D_i, q_i are the elastic displacements, electric potential, stresses, electric displacements, and heat flux components, respectively, and T is the temperature. Furthermore, c_{ijkl}, e_{lij}, ε_{ij} and λ_{ij} are the elastic moduli, piezoelectric constants, dielectric constants and the heat conduction coefficients, respectively. The values β_{iJ} are the stress–temperature coefficients for $J = 1$, 2, 3 and β_{i4} present the pyroelectric constants. Small subscripts in (1)–(5) and afterwards are always ranging from 1 to 3, capital subscripts are ranging from 1 to 4 and summation on repeated Latin suffixes has been used.

Assuming all fields are independent on the coordinate x_2 and using the method developed by Clements (1983) for thermoelastic problems, one obtains the following general solution to Eq. (2)

$$T = \chi'(z_t) + \bar{\chi}'(\bar{z}_t), \quad q_i = -(\lambda_{i1} + \tau\lambda_{i2})\chi''(z_t) - (\lambda_{i1} + \bar{\tau}\lambda_{i2})\bar{\chi}''(\bar{z}_t) \tag{6}$$

where $z_t = x_1 + \tau x_3$, the prime ($'$) denotes differentiation with respect to the argument, the overbar stands for the complex conjugate and τ is a root with a positive imaginary part of the equation

$$\lambda_{33}\tau^2 + (\lambda_{13} + \lambda_{31})\tau + \lambda_{11} = 0 \tag{7}$$

A general solution of Eq. (1) by using the Lekhnitskii–Eshelby–Stroh representation and its application to piezoelectric (Barnett and Lothe, 1975) and thermopiezoelectric (Shen and Kuang, 1998; Qin and Mai, 1999) materials can be presented in the form

$$\mathbf{V} = \mathbf{A}\mathbf{f}(z) + \mathbf{c}\chi(z_t) + \overline{\mathbf{A}\mathbf{f}}(\bar{z}) + \bar{\mathbf{c}}\bar{\chi}(\bar{z}_t) \tag{8}$$

$$\mathbf{t} = \mathbf{B}\mathbf{f}'(z) + \mathbf{d}\chi'(z_t) + \overline{\mathbf{B}\mathbf{f}}'(\bar{z}) + \bar{\mathbf{d}}\bar{\chi}'(\bar{z}_t) \tag{9}$$

where $z_J = x_1 + p_J x_3$, $\mathbf{V} = [u_1, u_2, u_3, \varphi]^{\mathrm{T}}$, $\mathbf{t} = [\sigma_{31}, \sigma_{32}, \sigma_{33}, D_3]^{\mathrm{T}}$ (the superscript 'T' stands for the transposed matrix), $\mathbf{A} = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4]$; p_J and $\mathbf{A}_J = [a_{1J}, a_{2J}, a_{3J}, a_{4J}]^{\mathrm{T}}$ are an eigenvalue and an eigenvector, respectively, of the system

$$\left\lfloor \mathbf{Q} + p_J(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p_J^2\mathbf{T} \right\rfloor \mathbf{A}_J = \mathbf{0} \tag{10}$$

with the elements of the 4×4 matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ defined as $Q_{JK} = E_{1JK1}$, $R_{JK} = E_{1JK3}$, $T_{JK} = E_{3JK3}$. The vector $\mathbf{c}$ is defined from the equation

$$\left\lfloor \mathbf{Q} + \tau(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + \tau^2\mathbf{T} \right\rfloor \mathbf{c} = \mathbf{N}_1 + \tau\mathbf{N}_2 \tag{11}$$

with $\mathbf{N}_m = [\beta_{m1}, \beta_{m2}, \beta_{m3}, \beta_{m4}]^{\mathrm{T}}$ ($m = 1$, 2) and the 4×4 matrix $\mathbf{B}$ and the vector $\mathbf{d}$ can be found by the formulas

$$\mathbf{B} = \mathbf{R}^{\mathrm{T}}\mathbf{A} + \mathbf{TAP}, \quad \mathbf{d} = (\mathbf{R}^{\mathrm{T}} + \tau\mathbf{T})\mathbf{c} - \mathbf{N}_2 \tag{12}$$

with $\mathbf{P} = \mathrm{diag}[p_1, p_2, p_3, p_3]$.

It is worth to mention that $\chi(z_t)$ from the formula (6) is an arbitrary analytic function and $\mathbf{f}(z)$ in the formulas (8) and (9) is an arbitrary analytic vector function with four components which should be determined later.

3. A bimaterial thermopiezoelectric space with a mixed boundary conditions at the interface

A bimaterial composed of two different piezoelectric semi-infinite spaces $x_3 > 0$ and $x_3 < 0$ with thermomechanical properties defined by the matrices $E^{(1)}_{iJKl}$, $\lambda^{(1)}_{ij}$, $\beta^{(1)}_{iJ}$ and $E^{(2)}_{iJKl}$, $\lambda^{(2)}_{ij}$, $\beta^{(2)}_{iJ}$, respectively, is considered (Fig. 1). We assume, that the component q_3 of the temperature flux vector and the vector $\mathbf{t}$ are continuous across the whole bimaterial interface and the parts $L_t = \{(-\infty, d_1) \cup (a_1, d_2) \cup \cdots (d_n, \infty)\}$ and $L = \{(-\infty, c_1) \cup (b_1, c_2) \cup \cdots (b_n, \infty)\}$ $([d_i, a_i] \subset [c_i, b_i])$ of the interface $-\infty < x_1 < \infty$, $x_3 = 0$ are thermally and electromechanically bounded, respectively, i.e. the boundary conditions at the interface $x_3 = 0$ are the following

$$q_3^{(1)} = q_3^{(2)}, \quad \mathbf{t}^{(1)}(x_1, 0) = \mathbf{t}^{(2)}(x_1, 0) \quad \text{for } x_1 \in (-\infty, \infty) \tag{13}$$

$$T^{(1)} = T^{(2)} \quad \text{for } x_1 \in L_t, \quad \mathbf{V}^{(1)}(x_1, 0) = \mathbf{V}^{(2)}(x_1, 0) \quad \text{for } x_1 \in L \tag{14}$$

Presenting $(6)_2$ for $i = 3$ in the form

$$q_3 = -\mathrm{i}k\chi''(z_t) + \mathrm{i}k\overline{\chi}''(\bar{z}_t) \tag{15}$$

with $k = \lambda_{33}(\tau - \bar{\tau})/(2\mathrm{i})$ the heat flux continuity condition gives

$$\mathrm{i}k^{(1)}\chi_1''(x_1^+) + \mathrm{i}k^{(2)}\overline{\chi}_2''(x_1^+) = \mathrm{i}k^{(2)}\chi_2''(x_1^-) + \mathrm{i}k^{(1)}\overline{\chi}_1''(x_1^-) \tag{16}$$

where the signs "+" and "−" denotes the upper and lower parts of the interface, $\mathrm{i} = \sqrt{-1}$, and $k^{(1)}$, $\chi_1(z_t)$ and $k^{(2)}$, $\chi_2(z_t)$ are related to the upper and lower half-spaces, respectively. Introducing a new function

$$\chi^*(z_t) = \begin{cases} \mathrm{i}k^{(1)}\chi_1''(z_t) + \mathrm{i}k^{(2)}\overline{\chi}_2''(z_t), & \text{for } x_3 > 0 \\ \mathrm{i}k^{(2)}\chi_2''(z_t) + \mathrm{i}k^{(1)}\overline{\chi}_1''(z_t), & \text{for } x_3 < 0 \end{cases} \tag{17}$$

and using Eq. (16) one can see that the function $\chi^*(z)$ is analytic in the whole plane. Assuming the heat flux disappears at infinity and by using Liouville's theorem one obtains that $\chi^*(z) \equiv 0$ and the formula (17) leads to

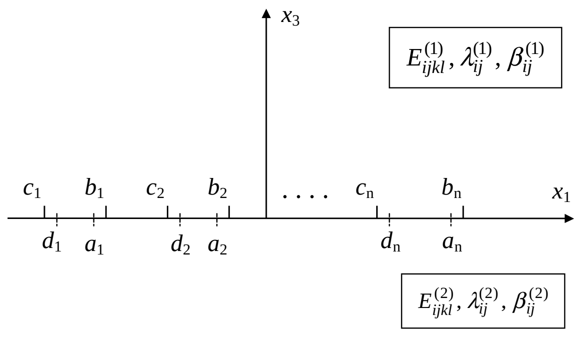

Fig. 1. Bimaterial thermopiezoelectric plane with mixed boundary conditions.

$$\overline{\chi}_2''(z_t) = -\frac{k^{(1)}}{k^{(2)}}\chi_1''(z_t) \quad \text{for } x_3 > 0, \qquad \overline{\chi}_1''(z_t) = -\frac{k^{(2)}}{k^{(1)}}\chi_2''(z_t) \quad \text{for } x_3 < 0 \tag{18}$$

Further, by introducing the derivative of the temperature jump across the bimaterial interface

$$[T'(x_1)] = T_1'^{+}(x_1, 0) - T_2'^{-}(x_1, 0) \tag{19}$$

and by using Eqs. $(6)_1$, (18) leads to

$$[T'(x_1)] = \theta''^{+}(x_1) - \theta''^{-}(x_1) \tag{20a}$$

where

$$\theta(z_t) = \begin{cases} (1 + k^{(1)}/k^{(2)})\chi_1(z_t), & \text{for } x_3 > 0 \\ (1 + k^{(2)}/k^{(1)})\chi_2(z_t), & \text{for } x_3 < 0 \end{cases} \tag{20b}$$

and the formula (15) written for the upper part of the interface as well as Eqs. (18) and (20b) lead to

$$q_2^{(1)}(x_1, 0) = -\mathrm{i}k_0\{\theta''^{+}(x_1) + \theta''^{-}(x_1)\} \tag{21a}$$

with

$$k_0 = \frac{k^{(1)}k^{(2)}}{k^{(1)} + k^{(2)}} \tag{21b}$$

It is clearly seen from Eqs. $(14)_1$, (19) and (20a) that the function $\theta(z_t)$ is analytic in the whole plane with a cut along $(-\infty, \infty) \setminus L_t$.

Using an approach developed for a thermoelastic case by Clements (1983) and the relations (9) and $(13)_2$ the following expressions at the interface are obtained

$$[\mathbf{V}'(x_1)] = \mathbf{W}^{+}(x_1) - \mathbf{W}^{-}(x_1) \tag{22}$$

$$\mathbf{t}^{(1)}(x_1, 0) = \mathbf{G}\mathbf{W}^{+}(x_1) - \overline{\mathbf{G}}\mathbf{W}^{-}(x_1) - \mathbf{g}(x_1) \tag{23}$$

where

$$[\mathbf{V}'(x_1)] = \mathbf{V}'^{(1)}(x_1, 0) - \mathbf{V}'^{(2)}(x_1, 0) \tag{24}$$

$\mathbf{G} = \mathbf{B}^{(1)}\mathbf{D}^{-1}$, $\mathbf{D} = \mathbf{A}^{(1)} - \overline{\mathbf{L}}\mathbf{B}^{(1)}$, $\mathbf{L} = \mathbf{A}^{(2)}(\mathbf{B}^{(2)})^{-1}$, $\mathbf{W}^{\pm}(x_1) = \mathbf{W}(x_1 \pm 0)$ and the vector-function $\mathbf{g}(x_1) = [g_1(x_1), g_2(x_1), g_3(x_1), g_4(x_1)]^{\mathrm{T}}$ can be presented in the form

$$\mathbf{g}(x_1) = \mathbf{h}\theta'^{+}(x_1) - \bar{\mathbf{h}}\theta'^{-}(x_1) \tag{25}$$

with

$$\mathbf{h} = \frac{1}{k^{(1)} + k^{(2)}}\{-\mathbf{G}(\overline{\mathbf{L}}\mathbf{d}^{*} - \mathbf{c}^{*}) - k^{(2)}\mathbf{d}^{(1)}\} \tag{26a}$$

and

$$\mathbf{c}^{*} = k^{(2)}\mathbf{c}^{(1)} + k^{(1)}\bar{\mathbf{c}}^{(2)}, \quad \mathbf{d}^{*} = k^{(2)}\mathbf{d}^{(1)} + k^{(1)}\bar{\mathbf{d}}^{(2)} \tag{26b}$$

It is worth to note that for the boundary conditions (14) the vector function $\mathbf{W}(z) = [W_1(z), W_2(z), W_3(z), W_3(z)]^{\mathrm{T}}$ is analytic in the whole plane with a cut along $(-\infty, \infty) \setminus L$. We note as well that the matrix $\mathbf{G}$ and the vector function $\mathbf{W}(z)$ are related to the matrix $\mathbf{H}$ and the vector function $\mathbf{\Psi}'(z)$ of the papers by Suo et al. (1992) and Shen and Kuang (1998) as $\mathrm{i}\mathbf{G}^{-1} = \mathbf{H}$, $\mathbf{W}(z) = -\mathrm{i}\mathbf{H}\mathbf{\Psi}'(z)$, respectively, and the relations (22) and (23) can be written without any difficulties in terms of the matrix $\mathbf{H}$ and the vector function $\mathbf{\Psi}'(z)$ from these papers. But for the formulation of the problems considered in the following chapters the presentation (22) and (23) appears to be more convenient than the form used in the mentioned papers. On the base of the relations (22) and (23) different problems of linear relationship can be formulated for thermopiezoelectric bimaterials with cuts at the material interfaces.

The attention is focused in the following on thermopiezoelectric materials of the symmetry class 6mm (Parton and Kudryavtsev, 1988) poled in the direction x_3 which have an essential practical significance as so-called poled ceramics. In this case for all fields which are independent of the coordinate x_2 the displacement V_2 of the vector-function $\mathbf{V}$ of Eq. (3) decouples in the (x_1, x_3)-plane from the components (V_1, V_3, V_4). Because of the simplicity of the V_2-determination our attention will be focused on the plane problem for the components (V_1, V_3, V_4). In this case similarly to the contracted notations in the anisotropic elasticity (Sokolnikoff, 1956) the following relations for the elements of the matrix $\mathbf{E}$ related to the (x_1, x_3)-plane can be introduced: $E_{1111} = c_{11}$, $E_{1133} = c_{13}$, $E_{3333} = c_{33}$, $E_{1313} = c_{44}$, $E_{1143} = e_{31}$, $E_{3343} = e_{33}$, $E_{1341} = e_{15}$, $E_{1441} = -\varepsilon_{11}$, $E_{3443} = -\varepsilon_{33}$. Moreover, the matrix $\mathbf{G}$ without the second row and column and the vector $\mathbf{h}$ without the second element have the following structure (Herrmann and Loboda, 2000)

$$\mathbf{G} = \begin{bmatrix} G_{11} & G_{13} & G_{14} \\ G_{31} & G_{33} & G_{34} \\ G_{41} & G_{43} & G_{44} \end{bmatrix} = \begin{bmatrix} \mathrm{i}g_{11} & g_{13} & g_{14} \\ g_{31} & \mathrm{i}g_{33} & \mathrm{i}g_{34} \\ g_{41} & \mathrm{i}g_{43} & \mathrm{i}g_{44} \end{bmatrix}, \quad \mathbf{h} = \begin{Bmatrix} \mathrm{i}\theta_1 \\ \theta_3 \\ \theta_4 \end{Bmatrix} \tag{27}$$

where all g_{ij} and θ_i are real.

4. Formulation of the problem and the thermal solution

Consider now the same bimaterial as in the previous chapter and assume that a tunnel interface crack is situated in the region $c \leqslant x_1 \leqslant b$, $x_3 = 0$. The half-spaces are loaded at infinity with uniform stresses $\sigma_{33}^{(m)} = \sigma$, $\sigma_{13}^{(m)} = \tau$ and $\sigma_{11}^{(m)} = \sigma_{xm}^{\infty}$ as well as with uniform electric displacements $D_3^{(m)} = d$, $D_1^{(m)} = D_{xm}^{\infty}$ which satisfy the continuity conditions at the interface. Besides a uniform temperature flux q_0 in the x_3-direction is imposed at infinity. Because the load and the temperature flux do not depend on the coordinate x_2 the displacements $u_k^{(m)}$ are independent on x_2 and for the matrix $\mathbf{G}$ in the form (27) a two-dimensional problem in the (x_1, x_3)-plane can be considered (Fig. 2).

It will be assumed that the crack surfaces are traction-free for $x_1 \in [c, a] = L_1$ whilst they should be in frictionless contact for $x_1 \in (a, b) = L_2$, and the position of the point a is arbitrarily chosen for the time

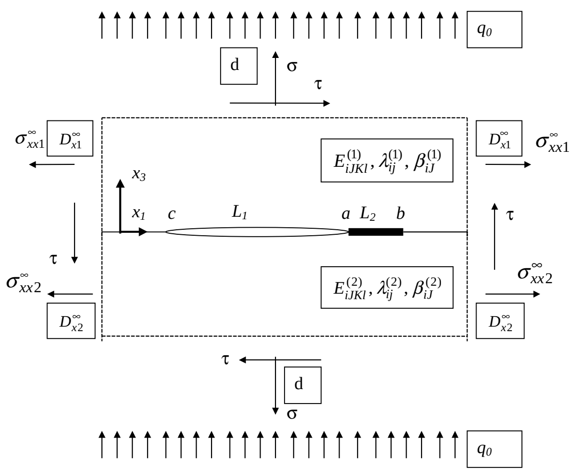

Fig. 2. Thermopiezoelectric plane with an interface crack under thermoelectromechanical loading.

being. This formulation is valid for the case when a longer contact zone arises at the right crack tip. In this case according to the results of Dundurs and Gautesen (1988) the oscillating singularity at the left crack tip will not significantly influence the stress and strain fields at the right crack tip. If the load calls a longer contact zone at the left crack tip, then it can be taken into account by a simple transposition of the half-spaces. The open part of the crack is thermally insulated, whereas ideal thermal contact takes place on the other part of the interface, and the electrical potential is assumed to be continues across the whole interface.

Taking into account the last assumption and combining each group of Eqs. (22) and (23) in the same way as it has been performed by Herrmann and Loboda (2000) one arrives at the following presentations:

$$\sigma_{33}^{(1)}(x_1,0) + \mathrm{i}m_j\sigma_{13}^{(1)}(x_1,0) = \vartheta_j[F_j^+(x_1) + \gamma_j F_j^-(x_1)] + \sigma_0 - g_{0j}(x_1), \quad (j=1,3) \tag{28}$$

$$[u_1'(x_1)] + \mathrm{i}S_j[u_3'(x_1)] = F_j^+(x_1) - F_j^-(x_1) \tag{29}$$

where

$$F_j(z) = W_1(z) + \mathrm{i}S_jW_3(z), \quad g_{0j}(x_1) = g_3(x_1) + \mathrm{i}m_jg_1(x_1) \tag{30a}$$

$$\gamma_j = -(g_{31} + m_jg_{11})/\vartheta_j, \quad S_j = \frac{g_{33} + m_jg_{13}}{g_{31} - m_jg_{11}}, \quad \vartheta_j = g_{31} - m_jg_{11}, \quad j=1,3 \tag{30b}$$

$$m_{1,2} = \mp\sqrt{-\frac{g_{31}g_{33}}{g_{11}g_{13}}}$$

and

$$\sigma_0 = -g_{34}\Delta_1^{-1}(g_{43}\sigma - g_{33}d), \quad \Delta_1 = g_{33}g_{44} - g_{43}g_{34} \tag{30c}$$

It follows from the first equation of (30a) that the functions $F_j(z)$ ($j = 1, 3$) under the boundary conditions $(14)_2$ are analytic in the whole plane with a cut along $(-\infty,\infty)\setminus L$. Besides, according to Herrmann and Loboda (2000) the function $W_4(z)$ is analytic in the whole plane, and this function can be excluded from the following consideration by means of incorporating the constant σ_0 in Eq. (28).

Because of the linearity the formulated problem can be considered separately for an electromechanical and a thermal loading, respectively. Taking into account that the problem in question under a pure electromechanical loading has been already studied in detail by Herrmann and Loboda (2000) in the following thermal loading only will be considered assuming $\sigma = \tau = \sigma_{xxm}^{\infty} = 0$ and $d = D_{xm}^{\infty} = 0$ for the time being. Moreover, the solution of the obtained problem can be constructed as the sum of two parts—a state of uniform heat flux q_0 and a perturbed temperature field caused by the insulated crack faces that tends to zero at a large distance from the crack. Because the homogeneous temperature field is out of our interest we are considering the perturbed state with the continuity and boundary conditions at the interface in the following form:

$$[T] = 0, \quad [q_3] = 0 \quad \text{for } x_1 \in L_t = (-\infty,\infty)\setminus(c,a) \tag{31a}$$

$$q_3^{\pm} = -q_0 \quad \text{for } x_1 \in L_1 \tag{31b}$$

$$[\mathbf{V}(x_1,0)] = 0, \quad [\mathbf{t}(x_1,0)] = 0 \quad \text{for } x_1 \in L \tag{32a}$$

$$\sigma_{13}^{(m)}(x_1,0) = 0, \quad \sigma_{33}^{(m)}(x_1,0) = 0, \quad [\varphi(x_1,0)] = 0, \quad [D_3(x_1,0)] = 0 \quad \text{for } x_1 \in L_1 \tag{32b}$$

$$\begin{aligned}&[u_3(x_1,0)] = 0, \quad [\varphi(x_1,0)] = 0, \quad \sigma_{13}^{(m)}(x_1,0) = 0, \quad [\sigma_{33}(x_1,0)] = 0,\\ &[D_3(x_1,0)] = 0 \quad \text{for } x_1 \in L_2\end{aligned} \tag{32c}$$

This problem is a particular case of the problem considered in the previous section for $n = 1$, $c_1 = d_1 = c$, $a_1 = a$, $b_1 = b$ and therefore the presentations (20a), (21a) and (22), (23) obtained there hold true in this case.

Satisfying the boundary condition (31b) by using Eq. (21a) the following Hilbert problem arises

$$\theta''^{+}(x_1) + \theta''^{-}(x_1) = -\frac{\mathrm{i}q_0}{k_0} \quad \text{for } x_1 \in L_1 \tag{33}$$

The solution of this problem disappearing at infinity can be presented in the form of Muskhelishvili (1977)

$$\theta''(z) = -\frac{\mathrm{i}q_0}{k_0}\frac{X_0(z)}{2\pi\mathrm{i}}\int_{L_1}\frac{\mathrm{d}t}{X_0^{+}(t)(t-z)} \tag{34a}$$

with

$$X_0(z) = (z-c)^{-1/2}(z-a)^{-1/2} \tag{34b}$$

Evaluation of the integral (34a) leads to the formula

$$\theta''(z) = \frac{\mathrm{i}q_0}{2k_0}\left[\left(z - \frac{c+a}{2}\right)X_0(z) - 1\right] \tag{35}$$

which gives after integration the following expression

$$\theta'(z) = \frac{\mathrm{i}q_0}{2k_0}\left[\sqrt{(z-c)(z-a)} - \tilde{z}\right] \tag{36}$$

where $\tilde{z} = z - c_0$ and the integration constant $c_0 = (c+a)/2$ is introduced to satisfy the condition $\theta'(z) \to 0$ for $z \to \infty$.

By means of the obtained solution and the formulas (20a) and (21a) the temperature jump across the material interface for $c < x_1 < a$ and the temperature flux for $x_1 > a$ can be presented in the following form:

$$[T(x_1)] = -\frac{q_0}{k_0}\sqrt{(x_1-c)(a-x_1)}, \quad q_3^{(1)}(x_1,0) = q_0\left(\frac{x_1}{\sqrt{(x_1-c)(x_1-a)}} - 1\right) \tag{37}$$

The last two expressions completely define the temperature jump and the heat flux in the bimaterial for any position of point a.

5. On an admissible direction of the heat flux

For the case of isotropic materials the problem of a possible transition from a perfect thermal contact of two isotropic bodies to their separation has been discussed in the paper by Martin-Moran et al. (1983) where it was in particular indicated that such a possibility essentially depends on the material properties and on the direction of the heat flux. To the authors knowledge a similar analysis has not been performed yet neither for an anisotropic, nor for a piezoelectric bimaterial. Therefore to analyse the problem (31) and (32) correctly let us consider now an auxiliary problem concerning a possible transition from a perfect thermal contact of two piezoelectric bodies to their separation.

Consider the same piezoelectric half-spaces and the same loading as in Section 4 and assume that $\sigma < 0$ and $\tau = 0$, i.e. the half space are compressed to each other with a stress $\sigma_{33}^{(m)} = \sigma$ at infinity, and the thermal and the electrical fluxes are prescribed at infinity as well. The contact between the half-spaces is assumed to be frictionless and electrically permeable. A soft thermal insulator which does not call stresses is located in the region $|x_1| \leqslant a$ of the interface and the remaining part of the interface is in a perfect thermal contact.

Because the load does not depend on the coordinate x_2 the plane strain problem in the (x_1, x_3)-plane can be considered and the interface conditions for the perturbed thermal and the associated mechanical problems can be presented in the form

$$\text{for } |x_1| \leqslant a: \quad q_3^{\pm} = q_0, \quad \sigma_{13}^{(m)}(x_1, 0) = 0, \quad \sigma_{33}^{(m)}(x_1, 0) = 0, \quad [\varphi(x_1, 0)] = 0, \quad [D_3(x_1, 0)] = 0 \tag{38}$$

$$\begin{aligned} \text{for } |x_1| > a: \quad & [T] = 0, \quad [q_3] = 0, \quad [u_3(x_1, 0)] = 0, \quad [\varphi(x_1, 0)] = 0, \\ & \sigma_{13}^{(m)}(x_1, 0) = 0, \quad [\sigma_{33}(x_1, 0)] = 0, \quad [D_3(x_1, 0)] = 0 \end{aligned} \tag{39}$$

Thus, we assume that in the interval $|x_1| \leqslant a$ a crack with thermally insulated and mechanically free surfaces appears and that the other part of the material interface is in mechanically frictionless and thermally perfect contact. The possibility of such a formulation, providing an electrically permeable interface exists for $|x_1| < \infty$, is discussed in this paragraph.

Introducing the vectors $\mathbf{S} = [\sigma_{13}^{(1)}, [u_3'], [\varphi']]^{\mathrm{T}}$ and $\mathbf{P} = [[u_1'], \sigma_{33}^{(1)}, D_3^{(1)}]^{\mathrm{T}}$ the following relations can be found by means of the formulas (22) and (23)

$$\mathbf{S}(x_1) = \mathbf{M}\mathbf{W}^{+}(x_1) - \overline{\mathbf{M}}\mathbf{W}^{-}(x_1) + \mathbf{m}^{*}\theta'^{+}(x_1) - \bar{\mathbf{m}}^{*}\theta'^{-}(x_1) \tag{40}$$

$$\mathbf{P}(x_1) = \mathbf{N}\mathbf{W}^{+}(x_1) - \overline{\mathbf{N}}\mathbf{W}^{-}(x_1) + \mathbf{n}^{*}\theta'^{+}(x_1) - \bar{\mathbf{n}}^{*}\theta'^{-}(x_1) \tag{41}$$

where the matrices $\mathbf{M}$ and $\mathbf{N}$ and the vectors m^* and n^* have the following structure

$$\mathbf{M} = \begin{bmatrix} G_{11} & G_{13} & G_{14} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & 0 & 0 \\ G_{31} & G_{33} & G_{34} \\ G_{41} & G_{43} & G_{44} \end{bmatrix}, \quad \mathbf{m}^{*} = \begin{Bmatrix} -h_1 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{n}^{*} = \begin{Bmatrix} 0 \\ -h_3 \\ -h_4 \end{Bmatrix} \tag{42}$$

Further, by introducing a new vector-function $\boldsymbol{\Psi}(z) = [\Psi_1(z), \Psi_2(z), \Psi_3(z)]^{\mathrm{T}}$ by the following formula

$$\boldsymbol{\Psi}(z) = \begin{cases} \mathbf{M}\mathbf{W}(z) + \mathbf{m}^{*}\theta'(z), & \text{for } x_3 > 0 \\ \overline{\mathbf{M}}\mathbf{W}(z) + \bar{\mathbf{m}}^{*}\theta'(z), & \text{for } x_3 < 0 \end{cases} \tag{43}$$

one can get

$$\mathbf{S}(x_1) = \boldsymbol{\Psi}^{+}(x_1) - \boldsymbol{\Psi}^{-}(x_1) \tag{44a}$$

$$\mathbf{P}(x_1) = \mathbf{Q}\boldsymbol{\Psi}^{+}(x_1) - \overline{\mathbf{Q}}\boldsymbol{\Psi}^{-}(x_1) - \mathbf{e}\theta'^{+}(x_1) + \bar{\mathbf{e}}\theta'^{-}(x_1) \tag{44b}$$

with $\mathbf{Q} = \mathbf{N}\mathbf{M}^{-1}$ and $\mathbf{e} = \mathbf{Q}\mathbf{m}^{*} - \mathbf{n}^{*}$. It follows from the relation (44a) that under the boundary conditions (39) the vector function $\boldsymbol{\Psi}(z)$ is analytic in the whole plane with a cut along $(|x_1| \leqslant a, x_3 = 0)$.

The numerical analysis showed that for all analyzed combinations of materials the matrix $\mathbf{Q}$ is pure imaginary and the vector $\mathbf{e}$ is real, and therefore $\overline{\mathbf{Q}} = -\mathbf{Q}$, $\bar{\mathbf{e}} = \mathbf{e}$. It means that the relation (44b) can be written in the form

$$\mathbf{P}(x_1) = \mathbf{Q}\{\boldsymbol{\Psi}^{+}(x_1) + \boldsymbol{\Psi}^{-}(x_1)\} - \mathbf{e}\{\theta'^{+}(x_1) - \theta'^{-}(x_1)\} \tag{45}$$

Moreover, it follows from the relation (44a) and the equations $\sigma_{13}^{(1)}(x_1, 0) = 0$, $[\varphi(x_1, 0)] = 0$ for $|x_1| \leqslant a$ that the functions $\Psi_1(z)$ and $\Psi_3(z)$ are analytic in the whole plane and by use of the conditions at infinity one easily concludes that $\Psi_1(z) \equiv C_1$ and $\Psi_3(z) \equiv C_3$ (C_1, C_3 are arbitrary constants). The thermal solution in this case can be presented by the formulas (35)–(37) with $c = -a$ and the remaining boundary condition from Eq. (38) leads to the following equation

$$P_2(x_1) = 0 \quad \text{for } |x_1| \leqslant a \tag{46}$$

which involves the function $\Psi_2(x_1)$ only. Because the condition $[u_1'(x_1)]|_{x_1\to\infty} = 0$ is usually satisfied by means of an appropriate prescription of the stresses $\sigma_{11}^{(m)} = \sigma_{xm}^{\infty}$ at infinity one can choose the arbitrary constant $C_1 = 0$. Then the condition at infinity can be written by means of Eq. (45) as follows

$$2\begin{bmatrix} Q_{22} & Q_{23} \\ Q_{32} & Q_{33} \end{bmatrix}\begin{Bmatrix} \Psi_2(z) \\ C_3 \end{Bmatrix}_{z\to\infty} = \begin{Bmatrix} \sigma \\ d \end{Bmatrix} \tag{47a}$$

and leads to the following formulas

$$\Psi_2(z)|_{z\to\infty} = 0.5D_0^{-1}(Q_{33}\sigma - Q_{23}d) \tag{47b}$$

$$C_3 = -0.5D_0^{-1}(Q_{32}\sigma - Q_{22}d) \quad \text{with } D_0 = Q_{22}Q_{33} - Q_{23}Q_{32} \tag{47c}$$

By use of the formulas (36) and (45), Eq. (46) can be written in the form

$$\Psi_3^+(x_1) + \Psi_3^-(x_1) = \frac{e_2 q_0}{k_0 \operatorname{Im}(Q_{22})}\sqrt{x_1^2 - a^2} - 2\frac{Q_{23}}{Q_{22}}C_3 \quad \text{for } |x_1| \leqslant a \tag{48a}$$

The solution of the Riemann problem (48) under the condition (47b) at infinity can be written in a form similar to Eq. (34a) (see Muskhelishvili (1977) as well), which after an integration gives

$$\Psi_3(z) = \frac{e_2 q_0}{2\pi i k_0 \operatorname{Im}(Q_{22})}\left(\frac{2az}{\sqrt{z^2 - a^2}} + \sqrt{z^2 - a^2}\ln\frac{z-a}{z+a}\right) + \frac{\sigma}{Q_{22}}\frac{z}{\sqrt{z^2 - a^2}} - 2\frac{Q_{23}}{Q_{22}}C_3 \tag{48b}$$

The normal stress and the derivative of the normal displacement jump at the interface by use of the last formula and Eqs. (44a) and (45), read then as follows

$$\sigma_{33}^{(1)}(x_1, 0) = \frac{e_2 q_0}{\pi k_0}\left(\frac{2ax_1}{\sqrt{x_1^2 - a^2}} + \sqrt{x_1^2 - a^2}\ln\frac{x_1 - a}{x_1 + a}\right) + \frac{\sigma x_1}{\sqrt{x_1^2 - a^2}} \quad \text{for } x_1 > a \tag{49a}$$

$$[u_3'(x_1)] = [\operatorname{Im}(Q_{22})]^{-1}\left\{\frac{e_2 q_0}{\pi k_0}\left(-\frac{2ax_1}{\sqrt{a^2 - x_1^2}} + \sqrt{a^2 - x_1^2}\ln\frac{a - x_1}{a + x_1}\right) - \frac{\sigma x_1}{\sqrt{a^2 - x_1^2}}\right\} \quad \text{for } |x_1| < a \tag{49b}$$

It is worth to note that neither the normal stress nor the derivative of the normal displacement jump depend on the electrical flux d.

The formulas (49) are valid for any values of q_0 and σ, respectively, and it can be clearly seen that $\sigma_{33}^{(1)}(x_1, 0)$ and $[u_3'(x_1)]$ posses a square root singularity at the edges of the zone of thermal insulation. However, the most important situation concerning the investigation of the contact zone model for an interface crack is related to the case when the singularities in Eqs. (49) are absent. This situation can take place if the following relation between q_0 and σ is valid

$$\sigma = -\frac{2ae_2 q_0}{\pi k_0} \tag{50}$$

In this case the formulas (49) take the following form

$$\sigma_{33}^{(1)}(x_1, 0) = \frac{e_2 q_0}{\pi k_0}\sqrt{x_1^2 - a^2}\ln\frac{x_1 - a}{x_1 + a} \quad \text{for } x_1 > a \tag{51a}$$

$$[u_3'(x_1)] = [\operatorname{Im}(Q_{22})]^{-1}\frac{e_2 q_0}{\pi k_0}\sqrt{a^2 - x_1^2}\ln\frac{a - x_1}{a + x_1} \quad \text{for } |x_1| < a \tag{51b}$$

Because the logarithmic functions in (51a) and in (51b) are negative for points x_1 situated in the vicinity of point a and according to the numerical calculation the inequality $\operatorname{Im}(Q_{22}) > 0$ is valid, it follows that $\sigma_{33}^{(1)}(x_1, 0)$ and $[u_3'(x_1)]$ are negative in the vicinity of point a (whereas $[u_3(x_1)]$ is positive) if and only if

$$e_2 q_0 > 0 \tag{52}$$

holds true. The last inequality defines the direction of the temperature flux q_0 for which a transition from a perfect thermal contact of two piezoelectric bodies to their separation is possible. In spite of a particular problem is considered in this paragraph, nevertheless the inequality (52) can be treated as a general condition because the relations (51) define the asymptotic behavior of $\sigma_{33}^{(1)}(x_1, 0)$ and $[u_3'(x_1)]$, respectively, at the transition point a.

In terms of the matrix $\mathbf{G}$ and the vector $\mathbf{h}$ from Eq. (27) the inequality (52) can be written in the following form

$$\frac{g_{11}\theta_3 - g_{31}\theta_1}{g_{11}} q_0 > 0 \tag{53}$$

and for the case of two isotropic materials it is reduced to the well known inequality presented by Martin-Moran et al. (1983)

$$(\delta^{(1)} - \delta^{(2)}) q_0 > 0 \tag{54}$$

where $\delta^{(1)}$ and $\delta^{(2)}$ are the distortivities of the upper and lower materials, respectively. If the inequality (52) is not valid then the conditions (39) cannot be used and an imperfect thermal contact (Barber and Comninou, 1983) has to be considered. However, the consideration of an imperfect thermal contact is out of the subject of the present paper.

6. The classical interface crack model

Now we return to the consideration of the electromechanical characteristics of the problem (31) and (32). However, for the sake of comparison the classical interface crack model should be considered briefly when the crack is assumed to be completely opened, i.e. the contact zone L_2 disappears and the point a coincides with b. It is worth to remind as well that the perturbed thermal problem with zero mechanical loading at infinity is considered for the time being.

Satisfying the rest of the boundary conditions (32b) by means of Eq. (28) one arrives at the following Hilbert problems

$$F_j^+(x_1) + \gamma_j F_j^-(x_1) = g_{0j}(x_1)/\vartheta_j \quad \text{for } x_1 \in (c, a), \quad j = 1, 3 \tag{55}$$

Because of the properties expressed by the formulas (30b) the problem (55) for $j = 3$ can be easily obtained from the same problem for $j = 1$ and therefore the problem (55) can be considered for $j = 1$ only. Thus in the following the index j will be dropped out and Eq. (55) can be written as

$$F^+(x_1) + \gamma F^-(x_1) = \frac{q_0}{k_0\vartheta}\varphi_1(x_1) \quad \text{for } x_1 \in (c, a) \tag{56a}$$

where according to Eq. (27) for the ceramics of the class 6mm holds true

$$\varphi_1(x_1) = \mathrm{i}m\theta_1\tilde{x}_1 + \mathrm{i}\theta_3\sqrt{(x_1 - c)(x_1 - a)}, \quad \tilde{x}_1 = x_1 - c_0 \tag{56b}$$

The solution of the problem (56a) can be presented in the form (Muskhelishvili, 1977)

$$F(z) = \frac{q_0}{k_0\vartheta}\frac{X(z)}{2\pi\mathrm{i}}\int_{L_1}\frac{\varphi_1(t)\,\mathrm{d}t}{X^+(t)(t - z)} + X(z)P(z) \tag{57a}$$

where

$$X(z) = (z - c)^{-1/2+\mathrm{i}\varepsilon}(z - a)^{-1/2-\mathrm{i}\varepsilon} \tag{57b}$$

$\varepsilon = \ln\gamma/2\pi$, $P(z) = C_1 z + C_0$ and C_0, C_1 are arbitrary coefficients.

Using the formula (A.7) with $a = c$, $b = a$ and taking into account that for $f(x_1) = x_1/X^+(x_1)$ follows $k = -\gamma$ and for $f(x_1) = \sqrt{(x_1 - c)(x_1 - a)}/X^+(x_1)$ $k = \gamma$ holds true the following expression for $F(z)$ are obtained

$$F(z) = \frac{\mathrm{i}\theta_1 q_0}{k_0\vartheta(1+\gamma)}\{\tilde{z} - X(z)[d_{11}z^2 + d_{12}z + d_{13}]\} + \frac{\mathrm{i}\theta_3 q_0}{k_0\vartheta(1-\gamma)}\{\sqrt{(z-c)(z-a)} - X(z)[d_{21}z^2 + d_{22}z + d_{23}]\} + X(z)P(z) \tag{58a}$$

where $d_{11} = d_{21} = 1$, $d_{12} = d_{22} = -\mathrm{i}\varepsilon(a - c) - (a + c)$,

$$d_{13} = 0.25(a+c)^2 - 0.125(a-c)^2(1+4\varepsilon^2) + 0.5\mathrm{i}\varepsilon(a^2 - c^2), \quad d_{23} = d_{13} - 0.125(a-c)^2 \tag{58b}$$

The coefficients of $P(z)$ should be determined by the condition at infinity and the condition of the single-valuedness of the displacements which due to (29) can be written as

$$\int_c^a \{F^+(x_1) - F^-(x_1)\}\,\mathrm{d}x_1 = 0 \tag{59}$$

It follows from the absence of any loading at infinity that $C_1 = 0$. Using for the consideration of the condition (59) the approach based upon the formulas (A.1)–(A.4) with $F(z)$ instead of $f(z)$ and noting that the part $F(z)$ without the term connected with $P(z)$ tends to 0 for large $|z|$ as $\mathrm{O}(z^{-2})$ one can immediately conclude that $C_0 = 0$. The same result has been obtained by means of the direct consideration of Eqs. (58) and (59) (for $c = -a$) and by using the integrals (A.9).

The stresses and the jump of the displacements can be found by means of the formulas (28) and (29). Particularly for $x_1 > a$ the stresses can be presented in the form

$$\sigma_{33}^{(1)}(x_1,0) + \mathrm{i}m\sigma_{13}^{(1)}(x_1,0) = \mathrm{i}\theta_1\frac{q_0}{k_0}\{\tilde{x}_1 - X(x_1)[d_{11}x_1^2 + d_{12}x_1 + d_{13}]\} + \mathrm{i}\theta_3\frac{q_0}{k_0}\frac{1+\gamma}{1-\gamma}\{\sqrt{(x_1-c)(x_1-a)} - X(x_1)[d_{21}x_1^2 + d_{22}x_1 + d_{23}]\} - g_0(x_1) \tag{60}$$

The electrical flux in this region according to Herrmann and Loboda (2000) can be calculated directly via $\sigma_{33}^{(1)}(x_1,0)$ in the form

$$D_3^{(1)}(x_1,0) = g_{33}^{-1}g_{43}[\sigma_{33}^{(1)}(x_1,0) + g_3(x_1)] - g_4(x_1) \tag{61}$$

Introducing similarly to Rice (1988) the stress intensity factors (SIFs) by the formula

$$K_1 + \mathrm{i}mK_2 = (x_1 - a)^{\mathrm{i}\varepsilon}\sqrt{2\pi(x_1 - a)}[\sigma_{33}^{(1)}(x_1,0) + \mathrm{i}m\sigma_{13}^{(1)}(x_1,0)]_{x_1\to a+0} \tag{62}$$

and using Eq. (60) one arrives at the following expressions for the conjugating SIFs

$$K_1 - \mathrm{i}mK_2 = -\mathrm{i}(\gamma+1)\frac{q_0}{k_0}\frac{l\sqrt{\pi l}}{4\sqrt{2}}\mathrm{e}^{-\mathrm{i}\psi}[(1+2\mathrm{i}\varepsilon)^2(\eta_1 + \eta_3) - \eta_3] \tag{63}$$

where

$$\eta_1 = -\frac{m\theta_1}{\gamma+1}, \quad \eta_3 = \frac{\theta_{31}}{\gamma-1} \tag{64}$$

$\psi = \varepsilon \ln l$ and $l = b - c$ is the crack length. The energy release rate (ERR) can by found by using the formula (Herrmann and Loboda, 2000)

$$G = \Omega(K_1^2 + m^2 K_2^2) \tag{65}$$

with $\Omega = -[2\vartheta(1+\gamma)m]^{-1}$.

Particularly for a homogeneous piezoelectric material ($\gamma = 1$, $\varepsilon = 0$, $m = -1$) is valid

$$K_{\mathrm{I}} = 0, \quad K_{\mathrm{II}} = -\frac{q_0}{k_0}\frac{l\sqrt{\pi l}}{4\sqrt{2}}\theta_1 \tag{66}$$

7. An artificial contact zone for an interface crack

If the position of the point a does not coincide with b, then an artificial contact zone shown in Fig. 2 is considered to exist now. The contact zone for an arbitrary position of the point a will be called an artificial one. Further, a solution of the problem (31) and (32) for an interface crack with such a zone will be found, and the real contact zone length will be derived from this solution.

Satisfying by means of Eq. (28) with $\sigma_0 = 0$ the boundary conditions (32b) leads to Eq. (55) which for $j = 1$ can be written in the form (56a). The boundary conditions (32c) by using of Eqs. (28) and (29) lead to the following equations (the index j is dropped again on the same reason as earlier)

$$\mathrm{Im}\, F^{\pm}(x_1) = \frac{q_0}{k_0\vartheta}\varphi_2(x_1) \quad \text{for } x_1 \in L_2 \tag{67a}$$

where

$$\varphi_2(x_1) = \frac{m\theta_1}{1+\gamma}\left[\tilde{x}_1 - \sqrt{(x_1 - c)(x_1 - a)}\right] \tag{67b}$$

The relations (56) and (67) represent an inhomogeneous combined Dirichlet–Riemann boundary value problem for the sectionally-holomorphic functions $F(z)$. A solution of an associated homogeneous problem with respect to Eqs. (56) and (67) was found and applied to the analysis of a rigid stamp by Nahmein and Nuller (1986), and concerning the problem of an interface crack this solution has been developed by Loboda (1993) and Herrmann and Loboda (1999).

The general solution of the problem (56) and (67) can be presented in the form

$$F(z) = P(z)X_1(z) + Q(z)X_2(z) + F_0(z) \tag{68}$$

where $P(z) = C_1 z + C_2$, $Q(z) = D_1 z + D_2$ are polynomials with arbitrary real coefficients,

$$X_1(z) = \mathrm{i}\, \mathrm{e}^{\mathrm{i}\varphi(z)}/\sqrt{(z-c)(z-b)} \quad \text{and} \quad X_2(z) = \mathrm{e}^{\mathrm{i}\varphi(z)}/\sqrt{(z-c)(z-a)} \tag{69a}$$

are the particular solutions of a homogeneous problem correspondent to (56) and (67), with

$$\varphi(z) = 2\varepsilon \ln \frac{\sqrt{(b-a)(z-c)}}{\sqrt{l(z-a)} + \sqrt{(a-c)(z-b)}} \tag{69b}$$

A particular solution $F_0(z)$ of the inhomogeneous problem (56) and (67) is given in the Appendix B according to

$$F_0(z) = \frac{q_0}{k_0\vartheta}X_2(z)[\omega_1(z) + \omega_2(z)] \tag{70}$$

where $\omega_1(z)$, $\omega_2(z)$ are given by the formulas (B.3) and (B.7), respectively.

The form of the solution (68) and (70) is rather complicated and inconvenient for the determination of the arbitrary coefficients and the following use of this solution. Therefore, the obtained solution should be reduced to a more clear form. First of all it should be mentioned that the functions $X_m(z)$ $(m = 1, 2)$ have the following properties:

(i) they are analytic in the whole plane except the segment $[c, b]$;
(ii) for the upper "+" and the lower "−" sides of $[c, b]$ they have the following behavior

$$X_m(x_1) = X_m^+(x_1) = -\gamma X_m^-(x_1) \quad \text{for } x_1 \in (c, a) \tag{71a}$$

with

$$X_1^\pm(x_1) = \frac{\pm e^{\pm\varphi_0(x_1)}}{\sqrt{(x_1 - c)(b - x_1)}}, \quad X_2^\pm(x_1) = \frac{e^{\pm\varphi_0(x_1)}}{\sqrt{(x_1 - c)(x_1 - a)}} \quad \text{for } x_1 \in (a, b) \tag{71b}$$

where

$$\varphi_0(x_1) = 2\varepsilon \tan^{-1}\sqrt{\frac{(a-c)(b-x_1)}{(b-c)(x_1-a)}}(\varphi_0(x_1) = \mathrm{i}\varphi(x_1) \quad \text{for } x_1 \in (a, b)) \tag{71c}$$

The most important part of the particular solution is the function $\omega_1(z)$. The integral in (B.3) defining this function could not be evaluated in a closed form. But the most important imaginary part of $\omega_1(z)$ which is used in the formula (B.5) has been managed to evaluate analytically in the following way.

We present the function $\omega_1(z)$ in the following way

$$\omega_1(z) = m\theta_1\Phi_1(z) + \theta_3\Phi_2(z) \tag{72a}$$

where

$$\Phi_1(z) = \frac{1}{2\pi \mathrm{i}}\int_{L_1}\frac{\mathrm{i}\tilde{t}\,\mathrm{d}t}{X_2^+(t)(t-z)}, \quad \Phi_2(z) = \frac{1}{2\pi \mathrm{i}}\int_{L_1}\frac{\mathrm{i}\sqrt{(t-c)(t-a)}\,\mathrm{d}t}{X_2^+(t)(t-z)}, \quad \tilde{t} = t - c_0 \tag{72b}$$

Using the properties (71b) of the function $X_2(z)$ it can be seen that the function $S_1(z) = e^{-\mathrm{i}\varphi(z)} + e^{\mathrm{i}\varphi(z)}$ is analytic on the interval $[a, b]$ and hence it is analytic in the whole plane except the segment $[c, a]$. Considering the Cauchy integral

$$\Omega_1(z) = \frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{\mathrm{i}\tilde{\xi}\sqrt{(\xi-c)(\xi-a)}S_1(\xi)}{\xi - z}\,\mathrm{d}\xi, \quad \tilde{\xi} = \xi - c_0 \tag{73}$$

where

$$\Omega_1(z) = \mathrm{i}\tilde{z}\sqrt{(z-c)(z-a)}S_1(z) - c_p z^p - \cdots - c_0 \tag{74}$$

Γ is a contour surrounding the segment $[c, a]$ in a clockwise direction (Fig. 3), the point z remains outside of this contour and $c_p, \ldots, c_0$ have the same sense as in formula (A.2). Further, letting the contour Γ shrink into the interval $[c, a]$, by using that

$$S_1^-(t) = \gamma e^{-\mathrm{i}\varphi(t)} + \gamma^{-1} e^{\mathrm{i}\varphi(t)}, \quad e^{\pm \mathrm{i}\varphi(t)} = \gamma^{\pm 0.5} e^{\pm \mathrm{i}\varphi^*(t)} \quad \text{for } x_1 \in (c, a) \tag{75}$$

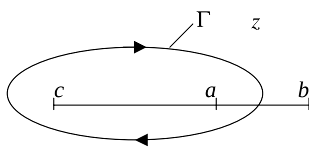

Fig. 3. A contour Γ surrounding the interval (c, a) of the jumps of the function $S_1(z)$ from the formula (73).

holds true, one gets the following relation:

$$\operatorname{Im}\Omega_1(x_1) = \frac{\gamma+1}{\pi\sqrt{\gamma}}\int_c^a \frac{t\sqrt{(t-c)(t-a)}\cos\varphi^*(t)}{t-x_1}\,\mathrm{d}t \quad \text{for } x_1 \notin (c,a) \tag{76}$$

where

$$\varphi^*(x_1) = 2\varepsilon\ln\frac{\sqrt{(b-a)(x_1-c)}}{\sqrt{l(a-x_1)}+\sqrt{(a-c)(b-x_1)}} \tag{77}$$

On the other hand by using the second of the relations (75) in the first of the integrals (72b) leads to the formula

$$\operatorname{Im}\Phi_1(x_1) = \frac{1}{2\pi\sqrt{\gamma}}\int_c^a \frac{t\sqrt{(t-c)(t-a)}\cos\varphi^*(t)}{t-x_1}\,\mathrm{d}t \quad \text{for } x_1 \notin (c,a) \tag{78}$$

A comparison of the formulas (76) and (78) leads to the relation

$$\operatorname{Im}\Phi_1(x_1) = \frac{1}{2(\gamma+1)}\operatorname{Im}\Omega_1(x_1) \tag{79}$$

Using the expansion for large $|z|$ of the function $\tilde{z}\sqrt{(z-c)(z-a)}S_1(z)$ and the formulas (A.2) and (79) gives the needed expression for $\operatorname{Im}\Phi_1(x_1)$. Applying a similar approach for the second integral in (72b) and combining the obtained results leads to the following expression for $\operatorname{Im}\omega_1(x_1)$ for $x_1 \notin (c,a)$

$$\operatorname{Im}\omega_1(x) = -\left\{\sqrt{(x-c)(x-a)}[\eta_1\tilde{x}+\eta_3\sqrt{(x-c)(x-a)}]\cos\varphi(x) + d_1x^2 + d_2x + d_3\right\} \tag{80a}$$

where

$$d_1 = -(\eta_1+\eta_3)\cos\beta, \quad d_2 = (\eta_1+\eta_3)[\beta_1\sin\beta + (a+c)\cos\beta],$$
$$d_3 = (\eta_1+\eta_3)\left\{\left(\frac{\beta_1^2}{2}-ac\right)\cos\beta + \beta_1\frac{b-3a-2c}{4}\sin\beta\right\} - \eta_3\frac{(a-c)^2}{8}\cos\beta \tag{80b}$$

with

$$\beta = 2\varepsilon\ln\frac{\sqrt{(b-a)}}{\sqrt{l}+\sqrt{(a-c)}}, \quad \beta_1 = \varepsilon\sqrt{(a-c)(b-c)} \tag{80c}$$

Substituting (80a) into (B.5) leads to

$$H^+(t) + H^-(t) = 2\{(\eta_1+\eta_3)(t-c)(t-a)\cosh\varphi_0(t) + d_1t^2 + d_2t + d_3\} \tag{81a}$$

$$H^+(t) - H^-(t) = 2\eta_1\{t\sqrt{(t-c)(t-a)} - (t-c)(t-a)\}\sinh\varphi_0(t) \tag{81b}$$

and the expression (B.7) attains a sufficiently clear form.

Moreover the determination of the arbitrary coefficients C_1, C_2 and D_1, D_2 from the formula (68) should be considered. Because thermoelectrical and mechanical fields are absent at infinity it immediately follows that $C_1 = D_1 = 0$. The coefficients C_2, D_2 can be found from the condition of the single-valuedness of the displacements which due to (29) can be written in the form

$$\int_c^b \{F^+(x_1) - F^-(x_1)\}\,\mathrm{d}x_1 = 0 \tag{82}$$

For a satisfaction of the condition (82) a way suggested in Appendix A by the formulas (A.1)–(A.4) will be used. In this case the segment $[c,b]$ should be used instead of $[a,b]$, $F(z)$ should be used as $f(z)$ and c_{-1} in this case is the coefficient before z^{-1} in the expansion of $F(z)$ at infinity. Taking into account that for large $|z|$

$$(t-z)^{-1} = -z^{-1} - tz^{-2} + \mathrm{O}(z^{-3}), \quad X_1(z) = \mathrm{i}z^{-2}\mathrm{e}^{\mathrm{i}\beta}\left(z + \mathrm{i}\beta_1 + \frac{c+b}{2}\right) + \mathrm{O}(z^{-3})$$

$$X_2(z) = z^{-2}\mathrm{e}^{\mathrm{i}\beta}\left(z + \mathrm{i}\beta_1 + \frac{c+a}{2}\right) + \mathrm{O}(z^{-3}) \text{ holds true}$$

it follows

$$c_{-1} = \left(\mathrm{i}C_2 + D_2 + \mathrm{i}\frac{q_0 R}{k_0 \vartheta}\right)\mathrm{e}^{\mathrm{i}\beta} \tag{83a}$$

where

$$R = \frac{1}{2\pi}\int_a^b \frac{H^+(t) + H^-(t)}{\sqrt{(t-a)(b-t)}}\,\mathrm{d}t \tag{83b}$$

is a real constant. Using the formulas (82) and (83a) and (A.4) lead to the results

$$C_2 = -(k_0\vartheta)^{-1} q_0 R, \quad D_2 = 0 \tag{84}$$

Evaluating the integral connected with d_1, d_2, d_3 in (83b) the expression for R can be presented in the form

$$R = \frac{\eta_1 + \eta_3}{\pi}\int_a^b (t-c)\sqrt{\frac{t-a}{b-t}}\cosh\varphi_0(t)\,\mathrm{d}t + R_0 \tag{85a}$$

where

$$R_0 = \left[\frac{(b+a)^2}{4} + \frac{(b-a)^2}{8}\right]d_1 + \frac{b+a}{2}d_2 + d_3 \tag{85b}$$

By using the obtained results the stresses and the jumps of the displacements can be calculated in accordance with the formulas (28) and (29). Particularly for $x_1 > b$ the formula for the stresses can be presented in the form

$$\sigma_{33}^{(1)}(x_1, 0) + \mathrm{i}m\sigma_{13}^{(1)}(x_1, 0) = (1+\gamma)\frac{q_0\,\mathrm{e}^{\mathrm{i}\varphi(x_1)}}{k_0\sqrt{x_1 - c}}\left[-\frac{\mathrm{i}R}{\sqrt{x_1 - b}} + \frac{\omega_1(x_1) + \omega_2(x_1)}{\sqrt{x_1 - a}}\right] - g_0(x_1) \tag{86}$$

where for $\omega_1(x_1)$ it is convenient to use the formula

$$\omega_1(x_1) = \frac{1}{2\pi\sqrt{\gamma}}\int_c^a \frac{\varphi_1(t)\sqrt{(t-c)(a-t)}\mathrm{e}^{-\mathrm{i}\varphi^*(t)}}{t - x_1}\,\mathrm{d}t$$

and the functions $\omega_2(x_1)$, $\varphi(x_1)$ and $\varphi^*(t)$ are real in the required intervals. For the determination of the electric flux $D_3^{(1)}(x_1, 0)$ in the region $x_1 > b$ the formula (61) with $\sigma_{33}^{(1)}(x_1, 0)$ defined by (86) can be used.

8. Stresses and electrical displacements at singular points

In this section the behavior of the stresses and the electrical displacements at the crack tip c and at the ends a, b of the artificial contact zone, respectively, should be considered. At the left crack tip c an oscillating singularity considered above takes place, and therefore no special attention to this point will be given. Further, it follows from the formula (86) that the normal stress is limited for $x_1 \to b+0$ and its value in the point $b+0$ read as follows

$$\sigma_{33}^{(1)}(b+0,0) = (1+\gamma)\frac{q_0}{k_0}\frac{\mathrm{Re}\,\omega_1(x_1)+\omega_2(x_1)}{\sqrt{l(b-a)}} - \mathrm{Re}\,g_0(x_1) \tag{87}$$

On the other hand the shear stress is singular for $x_1 \to b+0$ as well as the normal stress and the electrical flux for $x_1 \to a+0$. The following IFs are introduced to characterize these singularities

$$k_1 = \lim_{x_1 \to a+0} \sqrt{2\pi(x_1-a)}\sigma_{33}^{(1)}(x_1,0), \quad k_2 = \lim_{x_1 \to b+0} \sqrt{2\pi(x_1-b)}\sigma_{13}^{(1)}(x_1,0) \tag{88}$$

$$k_4 = \lim_{x_1 \to a+0} \sqrt{2\pi(x_1-a)}D_3^{(1)}(x_1,0)$$

Using for the determination of the SIF k_1 the formulas (28) and (68) and taking into account that $X_1(x_1)$, $g_0(x_1)$ are limited for $x_1 \to a+0$ as well as $Q(x_1) \equiv 0$, $\mathrm{e}^{\pm\varphi_0(a)} = \gamma^{\pm 0.5}$ are valid the following expression can be found

$$k_1 = \frac{q_0}{k_0}\sqrt{\frac{2\pi\gamma}{a-c}}\mathrm{Re}[2\omega_1(a+0)+\omega_2^+(a+0)+\omega_2^-(a+0)] \tag{89}$$

Applying the Plemeli formula (Muskhelishvili, 1977) and by using (B.3), (B.7), (B.8) and (81) leads to the following expression for the SIF

$$k_1 = \frac{q_0}{k_0}\sqrt{\frac{2}{\pi(a-c)}}(k_{11}+k_{12}) \tag{90a}$$

where

$$k_{11} = \int_c^a \sqrt{\frac{t-c}{a-t}}[-m\theta_1\tilde{t}\sin\varphi^*(t) + \theta_3\sqrt{(t-c)(a-t)}\cos\varphi^*(t)]\,\mathrm{d}t \tag{90b}$$

$$k_{12} = -2\eta_1\sqrt{\gamma}\int_a^b \left[(t-c) - \tilde{t}\sqrt{\frac{t-c}{t-a}}\right]\sinh\varphi_0(t)\,\mathrm{d}t \tag{90c}$$

It follows from the formula (86) that

$$k_2 = -(1+\gamma)\frac{q_0}{mk_0}\sqrt{\frac{2\pi}{l}}R \tag{91}$$

The SIF of the normal stress $k_1^b = \lim_{x_1 \to b-0}\sqrt{2\pi(b-x_1)}\sigma_{11}^{(1)}(x_1,0)$ at the point b can be presented by means of the formulas (28) and (68) via the SIF k_2 in the form

$$k_1^b = -m\frac{\gamma-1}{\gamma+1}k_2 \tag{92}$$

and the intensity factor of the electrical flux $(88)_3$ is defined completely by the SIF k_1 as it was already shown by Herrmann and Loboda (2000)

$$k_4 = g_{33}^{-1}\left[g_{43} - (g_{31}g_{43} - g_{41}g_{33})\frac{\gamma-1}{2\gamma T}\right]k_1 \tag{93}$$

It is clear from the last formulas that the SIFs k_1^b and k_4 are completely defined by the SIFs k_2 and k_1, respectively.

The energy release rates G_1^c, G_2^c related to the points a and b, respectively, according to Herrmann and Loboda (2000) can be presented in the form

$$G_1^c = -\frac{\cosh^2 \pi\varepsilon}{2m(1+\gamma)\vartheta} k_1^2, \quad G_2^c = -\frac{m}{2(1+\gamma)\vartheta} k_2^2 \tag{94}$$

For the following analysis it is convenient to introduce a parameter $\lambda = (b-a)/l$ defining a relative contact zone length. For any value of λ the SIFs k_1 and k_2 can be easily evaluated numerically from the formulas (90) and (91), respectively. Some difficulties can arise only for small values of the parameter λ when the behavior of the integrand at the boundaries of integration in (90b) becomes rather complicated. This case is very important because the real contact zone length is usually extremely small. Fortunately, for small values of λ the formulas (90) and (85a), (91) can be presented with high accuracy in a closed form.

In order to simplify (90b) this expression can be presented in the following form

$$k_{11} = 2\pi\sqrt{\gamma}\,\mathrm{Re}\,\omega_1(a+0) = 2\pi\sqrt{\gamma}\,\mathrm{Re}\left\{\frac{\mathrm{e}^{-\mathrm{i}\beta}}{2\pi\mathrm{i}}\int_c^a \frac{\varphi_1(t)\sqrt{(t-c)(t-a)}}{t-x}\mathrm{e}^{\mathrm{i}[\beta-\varphi(t)]}\,\mathrm{d}t\right\}_{x\to a+0} \tag{95}$$

Taking into account that

$$\beta - \varphi(t) = 2\varepsilon \ln \frac{\sqrt{(b-c)(t-a)} + \sqrt{(a-c)(t-b)}}{(\sqrt{b-c}+\sqrt{a-c})\sqrt{t-c}} \tag{96a}$$

holds true and using for a small λ the value $(a-c)$ instead of $(b-c)$ as well as the value $(t-a)$ instead of $(t-b)$ in this formula the expression (96a) can be rewritten as

$$\beta - \varphi(t) \approx \varepsilon \ln\left(\frac{t-a}{t-c}\right) \tag{96b}$$

and one arrives at the following expression:

$$k_{11} \approx 2\pi\sqrt{\gamma}\,\mathrm{Re}\lfloor \mathrm{e}^{-\mathrm{i}\beta} I(x)\rfloor_{x\to a+0} \tag{97a}$$

with

$$I(z) = \frac{1}{2\pi\mathrm{i}}\int_c^a \frac{\varphi_1(t)\,\mathrm{d}t}{X^+(t)(t-z)} \tag{97b}$$

and $X(z)$ defined by the formula (57b). An estimation of the performed simplification has been given by Herrmann and Loboda (2001). The integral (97b) has already been evaluated by the formula (58a) according to

$$I(a) = \mathrm{i}\frac{(a-c)^2}{8}\{(1-2\mathrm{i}\varepsilon)^2(\eta_1+\eta_3) - \eta_3\} \tag{98}$$

The SIFs k_1, k_2 for a small λ by assuming $(a-c) = l$, $(b-a) = 0$ can be presented in the form

$$\sqrt{\alpha}k_1 - \mathrm{i}mk_2 \approx \sqrt{\alpha}\tilde{k}_1 - \mathrm{i}m\tilde{k}_2 = -\mathrm{i}(\gamma+1)\frac{q_0}{k_0}\frac{l\sqrt{\pi l}}{4\sqrt{2}}\mathrm{e}^{\mathrm{i}\beta}\lfloor(1+2\mathrm{i}\varepsilon)^2(\eta_1+\eta_3) - \eta_3\rfloor \tag{99a}$$

where

$$\alpha = \frac{(\gamma+1)^2}{4\gamma} = \cosh^2(\pi\varepsilon) \tag{99b}$$

A numerical estimation had shown that these SIFs can be used with an admissible accuracy also for moderate values of λ.

Finally, we present the SIFs (88) for a pure electromechanical loading using the reference by Herrmann and Loboda (2000)

$$\sqrt{\alpha}k_1^{(\mathrm{em})} - \mathrm{i}mk_2^{(\mathrm{em})} = \sqrt{\frac{\pi l}{2}}(1+2\mathrm{i}\varepsilon)[\sqrt{1-\lambda}(\sigma\cos\beta + m\tau\sin\beta) + \mathrm{i}(\sigma\sin\beta - m\tau\cos\beta)] \tag{100}$$

which for small values of λ take the form

$$\sqrt{\alpha}\tilde{k}_1^{(\mathrm{em})} - \mathrm{i}m\tilde{k}_2^{(\mathrm{em})} = \sqrt{\frac{\pi l}{2}}\mathrm{e}^{\mathrm{i}\beta}(1+2\mathrm{i}\varepsilon)(\sigma - \mathrm{i}m\tau) \tag{101}$$

Superscript (em) is prescribed for the SIFs related to these loadings.

For the classical interface crack model the associated SIFs can be written as

$$K_1^{(\mathrm{em})} - \mathrm{i}mK_2^{(\mathrm{em})} = \sqrt{\frac{\pi l}{2}}\mathrm{e}^{-\mathrm{i}\psi}(1+2\mathrm{i}\varepsilon)(\sigma - \mathrm{i}m\tau) \tag{102}$$

and for a combination of electromechanical and thermal loading they must be found as a sum of the SIFs (63) and (102) in the classical case and (99), (100) or (101) by using a contact zone model. It is worth to note that the following relation between the SIFs of the classical and the contact zone (small λ) models is valid

$$(K_1 + K_1^{(\mathrm{em})}) - \mathrm{i}m(K_2 + K_2^{(\mathrm{em})}) = \mathrm{e}^{-\mathrm{i}(\psi+\beta)}[\sqrt{\alpha}(\tilde{k}_1 + \tilde{k}_1^{(\mathrm{em})}) - \mathrm{i}m(\tilde{k}_2 + \tilde{k}_2^{(\mathrm{em})})] \tag{103}$$

and moreover, the SIFs in the case of electrically permeable crack faces do not depend on the electrical flux d.

9. Contact zone model

The solution obtained in the Sections 7 and 8 is valid for any position of the point a. But this solution becomes physically correct if the following additional conditions

$$\sigma_{33}^{(1)}(x_1, 0) \leqslant 0 \quad \text{for } x_1 \in L_2, \quad [u_3(x,0)] \geqslant 0 \quad \text{for } x_1 \in L_1 \tag{104}$$

are satisfied. In this case a real contact zone in the sense of Comninou (1977) takes place at the crack tip. Assuming that the direction of the temperature flux satisfies the inequality (53) both inequalities (104) are satisfied when

$$k_1 + k_1^{(\mathrm{em})} = 0 \tag{105}$$

holds true at point a. For a large contact zone length this equation consisting of the formulas (90) and (100) can be solved numerically and a maximum root from the interval $(0, 1)$ should be taken. For a small value of λ the associated equation by using of (101) and (102) can be written in the following form

$$\mathrm{Re}\{\mathrm{e}^{\mathrm{i}\beta}(1+2\mathrm{i}\varepsilon)[1 - \mathrm{i}mk - p\mathrm{i}(m_3 + 2\mathrm{i}\varepsilon m_4)]\} = 0 \tag{106a}$$

where

$$k = \frac{\tau}{\sigma}, \quad m_3 = m_1 - \frac{4\varepsilon^2}{1+4\varepsilon^2}m_2, \quad m_4 = m_1 - \frac{2+4\varepsilon^2}{1+4\varepsilon^2}m_2$$

$$m_1 = \frac{-m\theta_1}{\vartheta k_0 \delta_2 (1+\gamma)}, \quad m_2 = \frac{\theta_3}{\vartheta k_0 \delta_2 (1-\gamma)} \tag{106b}$$

The dimensionless parameter p in Eq. (106a) having the following form

$$p = \frac{\gamma+1}{4}\frac{q_0 \vartheta \delta_2 l}{\sigma} \tag{107}$$

defines the intensity of the temperature flux q_0 with respect to the tensile stress σ at infinity. The parameter δ_2 was chosen to be equal to $\delta_2 = \beta_{11}^{(2)}/C_{66}^{(2)}\lambda_{11}^{(2)}(1 + (C_{13}^{(2)}/C_{11}^{(2)}))$, and it has the same dimension as the distortivity for an isotropic material.

Eq. (106a) can be rewritten in the following form

$$\mathrm{Re}\{e^{i\beta}(1 + 2i\varepsilon)(1 + ik^*)\} = 0 \tag{108a}$$

where the value

$$k^* = \frac{-mk - pm_3}{1 + 2p\varepsilon m_4} \tag{108b}$$

can be treated as a coefficient of the shear-normal loading with taking into account the intensity of the temperature flux. An exact solution of Eq. (108a) can be written in the form (Herrmann and Loboda, 1999)

$$\lambda_0 \approx \tilde{\lambda}_0 = 4\exp\left\{\frac{1}{\varepsilon}\left[\tan^{-1}\left(\frac{1 - 2\varepsilon k^*}{2\varepsilon + k^*}\right) + \pi n\right]\right\} \tag{109}$$

Here λ_0, $\tilde{\lambda}_0$ are the roots of Eqs. (105) and (108), respectively, and n is an integer number which should be used properly to define a maximum root in the interval $(0, 1)$ of Eq. (108a).

In the paper by Rice (1988) the following equation

$$\mathrm{Re}\{Kr_c^{i\varepsilon}/(1 + 2i\varepsilon)\} = 0 \tag{110}$$

has been suggested for the determination of the length r_c of the interpenetration zone of the crack faces predicted by the classical interface crack model. In spite of this equation has been suggested concerning isotropic materials it can be used for the considered problem as well because according to the results of Herrmann and Loboda (2000) the behavior of the displacement jumps at the crack tip of a permeable crack for a piezoelectric bimaterial can be defined by same formula as for an isotropic one provided the complex SIF is taken in the form

$$K = K_1 - imK_2 + K_1^{(em)} - imK_2^{(em)} \tag{111}$$

Using the expressions (63) and (102), Eq. (110) can be written in the form

$$\mathrm{Re}\{e^{i\varphi}(1 + ik^*)\} = 0 \tag{112a}$$

with

$$\varphi = \varepsilon \ln\left(\frac{r_c}{l}\right) \quad \text{and} \quad k^* \text{ taken from Eq. (108b)} \tag{112b}$$

The solution of Eq. (112a) can be easily given in a form similar to (109) and, moreover, it can be found from the comparison of Eqs. (108a) and (112a) the relation between the real contact zone length $\tilde{\lambda}_0 l$ and the length r_c of the interpenetration zone of the crack faces in the following form

$$\tilde{\lambda}_0 l = 4\exp[-\varepsilon^{-1}\tan^{-1}(2\varepsilon)]r_c \tag{113}$$

It is worth to remind that the relation (113) is valid for small values of (r_c/l) only.

The SIF $k_{20} + k_{20}^{(em)} = (k_2 + k_2^{(em)})|_{\lambda=\lambda_0}$ correspondent to a real contact zone length can be found by means of the formulas (85), (91) and (100) in which $\lambda = \lambda_0$ should be taken. For small values of λ_0 this SIF can be found by means of the following formula

$$\tilde{k}_{20} + \tilde{k}_{20}^{(em)} = \sigma\sqrt{\frac{\pi l}{2}}\mathrm{Im}\{e^{i\beta_0}(1 + 2i\varepsilon)(1 + ik^*)\} \tag{114a}$$

where

$$\beta_0 = \varepsilon \ln(0.25\tilde{\lambda}_0) \tag{114b}$$

10. Result and discussions

The formulas (108b) and (109) show that for a certain bimaterial and small values of λ_0 the contact zone length is completely defined by the parameter k^* only. The numerical analysis of several piezoelectric bimaterials had shown that the parameters m_1 and m_3 appeared to be always positive. Therefore, provided the direction of the temperature flux satisfies the inequality (53) and $\tau = 0$ holds true, by applying a tension σ and gradually increase of the heat flux, the contact zone length will get larger for $q_0\varepsilon > 0$ and smaller for $q_0\varepsilon < 0$. For a general tension-shear-thermal loading the situation is more complicated and various cases can arise. However, in the range of small λ_0 the increase of q_0 leads to a contact zone length defined by a limited value of the parameter k^* which is equal to

$$k^*_\infty = \lim_{q_0\to\infty} k^*(q_0) = -\frac{m_3}{2\varepsilon m_4} \tag{115}$$

It follows from Eq. (115) that for a small value of m_4 a long contact zone can occur for a pure thermal loading. However, in such cases a more correct verification of the contact zone length must be performed by means of Eq. (105).

The numerical results were obtained for a bimaterial composed of piezoelectric cadmium selenium (Ashida and Tauchert, 1997) (the upper material) and glass (the lower one). The characteristics of these materials are presented in Appendix C (column 1—the upper material, column 2—the lower material). Because for these materials the value of $(g_{11}\theta_3 - g_{31}\theta_1)/g_{11}$ appears to be negative the direction of the temperature flux q_0 according to the inequality (53) was chosen to be negative as well.

The variations of the value k^* which can be considered as the coefficient of normal-shear-thermal loading with respect to the intensity of the thermal flux p are shown in Fig. 4 for various coefficients of normal-shear loading k. It can be seen that for $k = 0$ ($\tau = 0$) the k^*-value increases with increasing intensity of the temperature flux p. On the other hand for $k = 10$ and $k = 50$ the k^*-values decrease with increasing p. However, in all three cases k^* tends for $p \to -\infty$ to the same value 0.810 which can be predicted by the formula (115).

In Fig. 5 the variation of the relative contact zone length λ_0 with respect to the same parameters as in Fig. 4 are shown. The values of λ_0 are usually extremely small, therefore, the logarithmic scale is used.

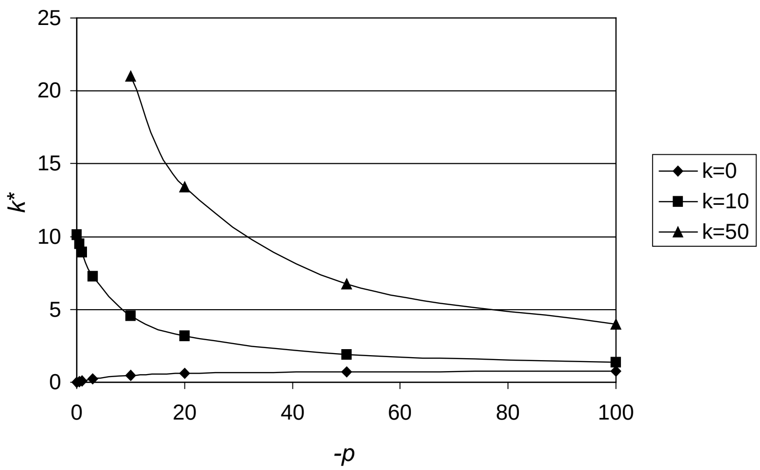

Fig. 4. The variation of the coefficient of normal-shear-thermal loading with respect to the intensity of the thermal flux p for different coefficients of normal-shear loading k.

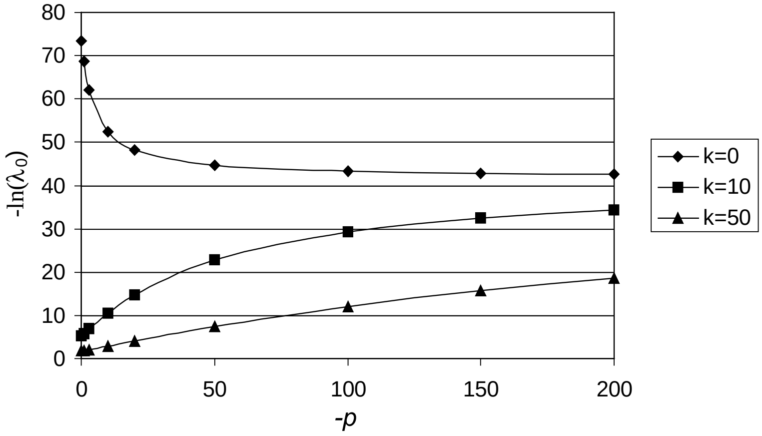

Fig. 5. The variation of the relative contact zone length λ_0 with respect to the intensity of the thermal flux p for different normal-shear loading coefficients k.

However for $k = 50$ and relatively small magnitudes of p the values of λ_0 are comparable to 1. This special case is demonstrated in the traditional scale in Fig. 6. For determination of λ_0 in Fig. 6, Eq. (105) has been used, and the differences of the correspondent results and the asymptotic values obtained by the formula (109) are shown in this figure. It is clear that for $\lambda_0 > 0.05$ this difference can be visualized, but for $\lambda_0 < 0.05$ it becomes negligible small and the formula (109) can be used for determination of λ_0. It should be noted that for any value of k the relative contact zone length λ_0 for $p \to -\infty$ tends to the same value 7.0225×10^{-19} as for a pure thermal loading.

Fig. 7 demonstrates the values of the SIF $k_{20}/(\sigma\sqrt{l})$ correspondent to the same parameters as in the Figs. 4 and 5. The SIF k_{20} grows with growing intensity of the heat flux, and of course its value essentially depends on the value of the shear loading τ defined by the parameter k.

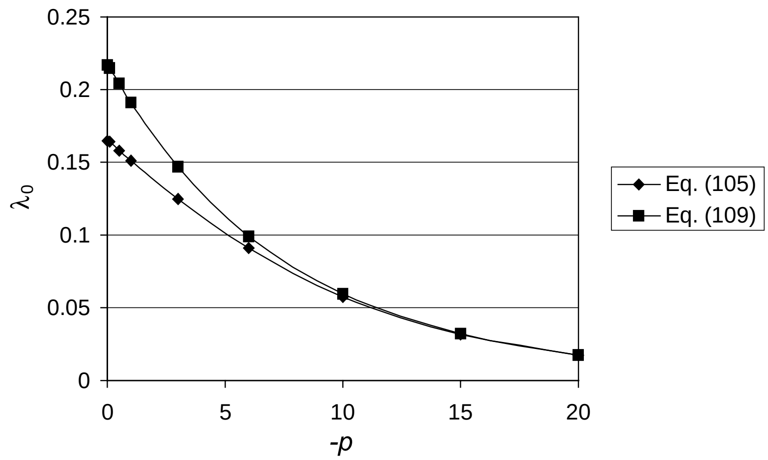

Fig. 6. The variation of the relative contact zone length λ_0 with respect to p for $k = 50$. Exact values obtained from Eq. (105) and asymptotic values given by Eq. (109) are presented.

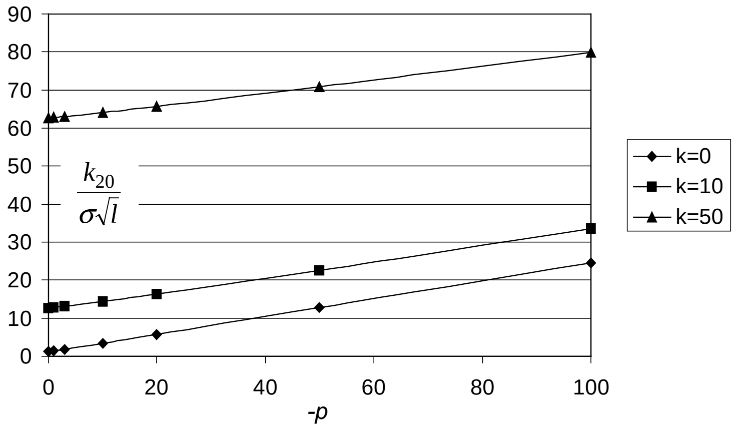

Fig. 7. The variation of the stress intensity factor k_{20} with respect to the intensity of the thermal flux p for different normal-shear loading coefficients k.

11. Conclusion

An interface crack between two semi-infinite piezoelectric spaces under the action of a combined thermoelectromechanical loading has been considered. By using the extended Lekhnitskii–Eshelby–Stroh representation developed for piezoelectric materials by Barnett and Lothe (1975) and Shen and Kuang (1998) scalar and matrix–vector expressions for the temperature flux and the derivatives of the temperature jump as well as for the stresses, electrical displacements and the derivatives of the mechanical displacement and electrical potential jumps at the interface via sectionally-holomorphic functions have been presented.

Furthermore, an auxiliary problem concerning an admissible direction of the temperature flux has been considered. It has been found that the inequality (52) defining the heat flux direction permits a transition from a perfect thermal contact of two piezoelectric bodies to their mechanical and thermal separation. This inequality can be treated as a generalization for piezoelectric bimaterials of the associated inequality presented for isotropic materials by Martin-Moran et al. (1983).

An open crack model is considered briefly and analytical expressions for the stresses, electrical flux, stress intensity factors and the energy release rate corresponding to this model have been given by the formulas (60)–(65).

The main attention in the paper has been devoted to the analysis of the contact of the interface crack faces. Assuming an existence of an artificial contact zone at the right crack tip an inhomogeneous combined Dirichlet–Riemann problem has been formulated. An exact analytical solution of this problem has been presented in the form of Cauchy type integrals and the stress and electrical displacement intensity factors have been expressed in the form of rather simple integrals (90) and (91). For a small contact zone length a closed form asymptotic formula (99) has been derived for the mentioned intensity factors. Moreover, a single analytical relation (103) between these intensity factors and the stress intensity factors of the classical model has been obtained.

The real contact zone length in a Comninou sense has been derived as a particular case of the obtained solution. Namely, a transcendental equation (105) for the determination of this length has been obtained. The solution of this equation consisting of rather simple integrals can be found numerically for any relative contact zone length λ_0. However, for small λ_0-values a closed form expression (109) has been found and k_{20} has been written in a rather easier form (114) as well.

Moreover, an important parameter k^* defined by formula (108b) has been found. This parameter can be treated as a coefficient of the shear-normal loading by taking into account a temperature flux. By means of this parameter the relative contact zone length λ_0 and the associated stress intensity factor k_{20} can easily be evaluated. Numerical results obtained both for small and long contact zone lengths, for different loading coefficients and various intensities of the temperature flux show that depending on these parameters the growth of the intensity of the temperature flux can lead both to an increase or a decrease of the contact zone length and mostly to an increase of the associated stress intensity factor k_{20}.

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Appendix A. Some important formulas related to Cauchy integrals

The consideration of the problem of linear relationship leads to the following general formulas by applying Cauchy integrals. Let a function $f(z)$ be analytic in the whole plane with a cut along the segment $[a, b]$ of the axis x, and the boundary values of f from the lower and upper sides of $[a, b]$ should be $f^+(x_1)$ and $f^-(x_1)$. Consider the Cauchy integral

$$\Omega(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - z} \, d\xi \tag{A.1}$$

where Γ is a contour surrounding the segment $[a, b]$ in a clockwise direction, and assume that the point z remains outside of this contour. Then according to the formula 70.3′ of Muskhelishvili (1977) for any z outside Γ holds true

$$\Omega(z) = f(z) - c_p z^p - \cdots - c_0 \tag{A.2}$$

where $f(z)|_{z\to\infty} = c_p z^p + \cdots + c_0 + c_{-1} z^{-1} + c_{-2} z^{-2} + \cdots$

If the contour Γ shrinks into $[a, b]$ it can be shown that

$$\Omega(z) = \frac{1}{2\pi i} \int_a^b \frac{f^+(t) - f^-(t)}{t - z} \, dt \tag{A.3}$$

The expansions of the left and right sides of (A.3) valid for large $|z|$ read Muskhelishvili (1977)

$$\Omega(z) = c_{-1} z^{-1} + O(z^{-2}), \quad \frac{1}{2\pi i} \int_a^b \frac{f^+(t) - f^-(t)}{t - z} \, dt = -z^{-1} \frac{1}{2\pi i} \int_a^b [f^+(t) - f^-(t)] \, dt + O(z^{-2})$$

A comparison of the coefficients connected with the term z^{-1} at the left and right sides of (A.2) leads to

$$\int_a^b [f^+(t) - f^-(t)] \, dt = -2\pi i c_{-1} \tag{A.4}$$

This formula plays an important role concerning the consideration of the single-valuedness of the displacements.

Further, let us consider the integral

$$I(z) = \frac{1}{2\pi i} \int_a^b \frac{f^+(t)}{t - z} \, dt \tag{A.5}$$

by assuming that the boundary values of f are related to each other as follows

$$f^-(x_1) = k f^+(x_1) \tag{A.6}$$

Considering the integral (A.1) again and by shrinking the contour Γ to $[a,b]$ it follows that

$$I(z)=\frac{1}{1-k}\Omega(z)=\frac{1}{1-k}\{f(z)-c_pz^p-\cdots-c_0\} \tag{A.7}$$

The obtained formula is similar to the formula (110.40) of Muskhelishvili (1977), but it is related to a more general class of functions than the associated formula just mentioned.

Important integrals can be evaluated on the base of formulas (A.5)–(A.7). Let for example $a=-b$ and $f(z)=(z+b)^{-1/2+\mathrm{i}\varepsilon}(z-b)^{-1/2-\mathrm{i}\varepsilon}$. In this case are $k=-\gamma^{-1}$ and $\gamma=\mathrm{e}^{2\pi\varepsilon}$. Taking into account that for large $|z|$ $f(z)=f^{\infty}(z)=z^{-1}+2\mathrm{i}\varepsilon bz^{-2}+(0.5-2\varepsilon^2)b^2z^{-3}+\cdots$, it valids $I(z)=\gamma(\gamma+1)^{-1}f(z)$. An expansion of the left and right sides of the last equation for large $|z|$ by use of formula (A.5) can be written in the form

$$\sum_{m=1}^{\infty}\left\{z^{-m}\frac{1}{2\pi\mathrm{i}}\int_a^b t^{m-1}f^+(t)\,\mathrm{d}t\right\}=f^{\infty}(z) \tag{A.8}$$

A comparison of the coefficients connected with the term z^{-m} ($m=1,2,3$) at the left and right sides of (A.8) and taking into account that $\cosh\pi\varepsilon=(\gamma+1)/(2\sqrt{\gamma})$ leads to the following formulas

$$\int_{-b}^{b}\frac{1}{\sqrt{b^2-t^2}}\left(\frac{b-t}{b+t}\right)^{\mathrm{i}\varepsilon}\mathrm{d}t=\frac{\pi}{\cosh\pi\varepsilon},\quad \int_{-b}^{b}\frac{t}{\sqrt{b^2-t^2}}\left(\frac{b-t}{b+t}\right)^{\mathrm{i}\varepsilon}\mathrm{d}t=-\frac{2\mathrm{i}\pi b\varepsilon}{\cosh\pi\varepsilon},$$
$$\int_{-b}^{b}\frac{t^2}{\sqrt{b^2-t^2}}\left(\frac{b-t}{b+t}\right)^{\mathrm{i}\varepsilon}\mathrm{d}t=\frac{\pi b^2(1-4\varepsilon^2)}{2\cosh\pi\varepsilon} \tag{A.9}$$

Appendix B. On a particular solution of an inhomogeneous Dirichlet–Riemann problem

Denoting by $X_2(z)$ a particular solution of a homogeneous problem associated with the problem (56) and (67), Eq. (56) can be written as

$$\frac{F^+(x_1)}{X_2^+(x_1)}-\frac{F^-(x_1)}{X_2^-(x_1)}=\frac{q_0}{k_0\vartheta}\frac{\varphi_1(x_1)}{X_2^+(x_1)}\quad \text{for } x_1\in L_1 \tag{B.1}$$

A particular solution of Eq. (B.1) reads according to Muskhelishvili (1977) is the following

$$\frac{F(z)}{X_2(z)}=\frac{q_0}{k_0\vartheta}[\omega_1(z)+\omega_2(z)] \tag{B.2}$$

where

$$\omega_1(z)=\frac{1}{2\pi\mathrm{i}}\int_{L_1}\frac{\varphi_1(t)\,\mathrm{d}t}{X_2^+(t)(t-z)} \tag{B.3}$$

and $\omega_2(z)$ is an arbitrary function analytic on L_1. Taking into account that $X_2^{\pm}(x_1)$ is real on L_2, Eq. (67) leads to

$$\mathrm{Im}\,\omega_2^{\pm}(x_1)=H^{\pm}(x_1)\quad \text{for } x_1\in L_2 \tag{B.4}$$

where

$$H^{\pm}(x_1)=\frac{\varphi_2^{\pm}(x_1)}{X_2^{\pm}(x_1)}-\mathrm{Im}\,\omega_1(x_1) \tag{B.5}$$

Introducing a new function $\omega_3(z)=-\mathrm{i}\omega_2(z)$, Eq. (B.4) can be written as

$$\mathrm{Re}\,\omega_3^{\pm}(x_1)=H^{\pm}(x_1)\quad \text{for } x_1\in L_2 \tag{B.6}$$

By using a particular solution of the Dirichlet problem (B.6) presented by the formula (46.25) of the reference by Gakhov (1966) the function $\omega_2(z)$ can be written as follows

$$\omega_2(z) = \frac{Y(z)}{2\pi} \int_{L_2} \frac{H^+(t) + H^-(t)}{Y^+(t)(t-z)} \mathrm{d}t + \frac{1}{2\pi} \int_{L_2} \frac{H^+(t) - H^-(t)}{t-z} \mathrm{d}t \tag{B.7}$$

with

$$Y(z) = \sqrt{(z-a)(z-b)} \tag{B.8}$$

Further, by applying the formula (B.2) a particular solution of the inhomogeneous problem (56) and (67) can be given in the form

$$F_0(z) = \frac{q_0}{k_0 \vartheta} X_2(z)[\omega_1(z) + \omega_2(z)] \tag{B.9}$$

Appendix C. Characteristics of the bimaterial

	The upper material (Cad Se)	The lower material (glass)
$c_{11} \times 10^{-10}$ (N/m^2)	7.41	5.88
$c_{33} \times 10^{-10}$ (N/m^2)	8.36	5.88
$c_{13} \times 10^{-10}$ (N/m^2)	3.93	1.47
$c_{44} \times 10^{-10}$ (N/m^2)	1.32	2.21
e_{31} (C/m^2)	−0.16	0.00
e_{15} (C/m^2)	−0.138	0.00
e_{33} (C/m^2)	0.347	0.00
$\varepsilon_{11} \times 10^{10}$ (C/V m)	0.825	0.885
$\varepsilon_{33} \times 10^{10}$ (C/V m)	0.902	0.885
λ_{11} (W/m K)	9.00	0.74
λ_{33} (W/m K)	9.00	0.74
$\beta_{11} \times 10^{-6}$ (N/m^2 K)	0.621	1.13
$\beta_{33} \times 10^{-6}$ (N/m^2 K)	0.551	1.13
$\beta_{34} \times 10^{6}$ (C/m^2 K)	−2.94	0.00

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